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Weakly nonlinear discrete multipoint boundary value problems

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Abstract

In this paper we study nonlinear, discrete, multipoint boundary value problems of the form

$$x(t+1) = A(t)x(t) + \epsilon f\big(t, x(t)\big)$$

subject to

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0.$$

We provide sufficient conditions for the existence of solutions and we present a qualitative analysis of the way the solutions depend on the parameter ϵ.

Keywords: Boundary value problems; Implicit Function Theorem; Lyapunov–Schmidt; Projection

1. Introduction

In this paper we consider weakly nonlinear discrete-time systems

$$x(t+1) = A(t)x(t) + \epsilon f\big(t, x(t)\big) \tag{1}$$

subject to multipoint boundary conditions of the form

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0. \tag{2}$$

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Throughout our discussion we assume that for each t, $A(t)$ is a nonsingular $n \times n$ matrix, f is a continuously differentiable map from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$, ϵ is a "small" real parameter, and each matrix B_k is $n \times n$.

We are primarily interested in systems at resonance; that is, problems where the corresponding linear homogeneous boundary value problem

$$x(t+1) = A(t)x(t) \tag{3}$$

subject to

$$B_0 x(0) + B_1 x(1) + \cdots + B_N x(N) = 0, \tag{4}$$

has nontrivial solutions.

We establish sufficient conditions for the solvability of (1)–(2) and we provide a qualitative analysis of the dependence of the solutions on the parameter ϵ. This analysis allows us to establish a connection between the solution sets of the nonlinear system (1)–(2) and the linear homogeneous boundary value problem (3)–(4).

We analyze (1)–(2) using the Lyapunov–Schmidt procedure. Our analysis is completely self contained; however we include references [3–6,10,14–16,18,20] for those who wish to see a more abstract approach as well as for readers interested in other applications.

The results presented here extend previous work of Rodriguez [19] where the boundary value problems are nonlinear perturbations of Sturm–Liouville scalar equations and of Rodriguez [16] who considered only endpoint boundary conditions. The present paper also complements other results in the area of discrete boundary value problems. The interested reader may consult [1,2, 7–9,11,17].

2. Preliminaries

We are concerned with the existence of solutions of

$$x(t+1) = A(t)x(t) + \epsilon f\big(t, x(t)\big)$$

subject to

$$B_0 x(0) + B_1 x(1) + \cdots + B_N x(N) = 0.$$

We analyze Eqs. (1)–(2) by using operators on finite dimensional sequence spaces. In order to do this, we introduce the following spaces:

$$X = \big\{\phi : \{0, 1, \ldots, N\} \to \mathbb{R}^n\colon\ B_0\phi(0) + B_1\phi(1) + \cdots + B_N\phi(N) = 0\big\}$$

and

$$Y = \big\{\psi : \{0, 1, \ldots, N-1\} \to \mathbb{R}^n\big\}.$$

We will use the supremum norm on each of these finite dimensional spaces. For x in X we define

$$\|x\| = \sup_{t=0,1,\ldots,N} \big|x(t)\big|$$

and for y in Y we define

$$\|y\| = \sup_{t=0,1,\ldots,N-1} \big|y(t)\big|,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. It is evident that with these norms, X and Y are Banach spaces. For $v = (v_1, v_2, \dots, v_m)$ in a product space of the form $V_1 \times V_2 \times \cdots \times V_m$ we will let $\|v\| = \sum_{i=1}^m \|v_i\|_{V_i}$ where $\|\cdot\|_{V_i}$ denotes the norm on V_i.

We define the linear map $L : X \to Y$ by

$$(Lx)(t) = x(t+1) - A(t)x(t)$$

and $F : X \to Y$ is given by

$$(Fx)(t) = f\big(t, x(t)\big).$$

From this it follows that x is a solution of (1)–(2) if and only if

$$Lx = \epsilon Fx. \tag{5}$$

Let

$$\Phi(t) = \begin{cases} I & \text{if } t = 0, \\ A(t-1)A(t-2)\dots A(0) & \text{for } t = 1, 2, \dots. \end{cases}$$

It is well known [2,11,13] that Φ is a fundamental matrix solution of $x(t+1) = A(t)x(t)$.

Proposition 2.1. *The solution space of the linear homogeneous boundary value problem* (3)–(4) *has the same dimension as* $\ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))$.

Proof. Clearly the solution space of (3)–(4) has the same dimension as $\ker(L)$, and

$$\begin{aligned} x \in \ker(L) \quad &\Longleftrightarrow \quad x(t+1) = A(t)x(t) \quad \text{and} \quad B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0 \\ &\Longleftrightarrow \quad \text{there exists } c \text{ in } \mathbb{R}^n \text{ such that } x(t) = \Phi(t)c \quad \text{and} \\ &\qquad\quad B_0\Phi(0)c + B_1\Phi(1)c + \cdots + B_N\Phi(N)c = 0 \\ &\Longleftrightarrow \quad x(t) = \Phi(t)c \quad \text{where } c \in \ker\big(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)\big). \qquad \square \end{aligned}$$

Since $\ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))$ has dimension less than or equal to n, there exist vectors $b_1, b_2, \dots, b_j$, $0 \leqslant j \leqslant n$, in $\mathbb{R}^n$ such that $\{b_1, b_2, \dots, b_j\}$ is a basis for $\ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))$.

Definition 2.1. $S(t)$ is the $n \times j$ matrix whose ith column is $\Phi(t)b_i$.

Corollary 2.1. *x is in the kernel of L if and only if $x(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}^j$.*

3. The case of invertible L

We show that if $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, then the boundary value problem

$$x(t+1) = A(t)x(t) + \epsilon f\big(t, x(t)\big)$$

subject to

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0$$

has a solution for sufficiently small ϵ.

Proposition 3.1. *L is invertible if and only if $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible. In this case, the bounded linear map $L^{-1} : Y \to X$ is defined by*

$$\big(L^{-1}h\big)(t) = -\Phi(t)D^{-1}\left[B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_N\Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right] + \Phi(t)\sum_{i=0}^{t-1}\Phi^{-1}(i+1)h(i)$$

where $D = B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$.

Proof. Let $h \in Y$. $Lx = h$ if and only if there exists an element x in X that satisfies

$$x(t+1) = A(t)x(t) + h(t) \quad \text{for } t = 0, 1, \ldots, N-1.$$

By the well-known Variation of Constants formula [2,11,13], $Lx = h$ if and only if

$$x(t) = \Phi(t)x(0) + \Phi(t)\sum_{i=0}^{t-1}\Phi^{-1}(i+1)h(i) \tag{6}$$

and

$$B_0x(0) + B_1x(1) + \cdots + B_Nx(N) = 0. \tag{7}$$

Combining Eqs. (6) and (7) we have $Lx = h$ if and only if

$$B_0x(0) + B_1\big(\Phi(1)x(0) + \Phi(1)\Phi^{-1}(1)h(0)\big) + \cdots + B_N\left(\Phi(N)x(0) + \Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right) = 0 \tag{8}$$

or equivalently

$$\big(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)\big)x(0) = -\left[B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_N\Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right]. \tag{9}$$

Notice that $Lx = h$ if and only if there exists $x(0) \in \mathbb{R}^n$ that satisfies Eq. (9). If we can find such an $x(0)$, then we define x as in Eq. (6) and we will have $Lx = h$. Observe that if $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible, then there is one and only one $x(0)$ that satisfies Eq. (9), which implies there is one and only one x that satisfies $Lx = h$, and thus L is invertible, with

$$\big(L^{-1}h\big)(t) = -\Phi(t)D^{-1}\left[B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_N\Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right] + \Phi(t)\sum_{i=0}^{t-1}\Phi^{-1}(i+1)h(i)$$

where $D = B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$.

Likewise if there is one and only one x that satisfies $Lx = h$, then there is one and only one $x(0)$ that satisfies Eq. (9), and thus $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ must be invertible. □

In order to use arguments based on the Implicit Function Theorem, we now establish that F is continuously Fréchet differentiable. Readers interested in Calculus on Banach Spaces may consult [10,12].

Proposition 3.2. *The map $F : X \to Y$ is continuously Fréchet differentiable and $DF(\phi) : X \to Y$ is given by*

$$\big(DF(\phi)\big)(h)(t) = \left(\frac{\partial f}{\partial x}\big(t, \phi(t)\big)\right)\big(h(t)\big).$$

Proof. We will first show that F is differentiable.

Let $\phi \in X$, and let $A : X \to Y$ be defined by

$$(Ah)(t) = \frac{\partial f}{\partial x}\big(t, \phi(t)\big)h(t).$$

Let $\epsilon > 0$. Since f is continuously differentiable, for each $t = 0, 1, \ldots, N-1$ there exists $\delta_t > 0$ such that

$$\big|h(t)\big| < \delta_t \quad \Rightarrow \quad \left|f\big(t, \phi(t) + h(t)\big) - f\big(t, \phi(t)\big) - \frac{\partial f}{\partial x}\big(t, \phi(t)\big)\big(h(t)\big)\right| < \epsilon\big|h(t)\big|.$$

Let $\|h\| < \delta$ where $\delta = \min_{t\in\{0,1,\ldots,N-1\}}\{\delta_t\}$. Then

$$\begin{aligned}
&\big\|F(\phi + h) - F(\phi) - Ah\big\| \\
&\quad = \sup_{t\in\{0,1,\ldots,N-1\}} \big|F(\phi + h)(t) - F(\phi)(t) - (Ah)(t)\big| \\
&\quad = \sup_{t\in\{0,1,\ldots,N-1\}} \left|f\big(t, \phi(t) + h(t)\big) - f\big(t, \phi(t)\big) - \frac{\partial f}{\partial x}\big(t, \phi(t)\big)\big(h(t)\big)\right| < \epsilon\|h\|.
\end{aligned}$$

Therefore

$$\frac{\|F(\phi + h) - F(\phi) - Ah\|}{\|h\|} \leqslant \epsilon,$$

and thus F is differentiable with

$$\big(DF(\phi)\big)(h)(t) = \left(\frac{\partial f}{\partial x}\big(t, \phi(t)\big)\right)\big(h(t)\big).$$

We now must show that F is continuously differentiable. Let $\phi \in X$ and let $\epsilon > 0$.

Since f is continuously differentiable, for each $t \in \{0, 1, \ldots, N-1\}$ there exists $\delta_t > 0$ such that

$$\big|\phi(t) - \psi(t)\big| < \delta_t \quad \Rightarrow \quad \left|\frac{\partial f}{\partial x}\big(t, \phi(t)\big) - \frac{\partial f}{\partial x}\big(t, \psi(t)\big)\right| < \epsilon.$$

Let $\delta = \min_{t\in\{0,1,\ldots,N-1\}}\{\delta_t\}$.

Then

$$\|\phi - \psi\| < \delta \quad \Rightarrow \quad \left|\frac{\partial f}{\partial x}\big(t, \phi(t)\big) - \frac{\partial f}{\partial x}\big(t, \psi(t)\big)\right| < \epsilon \quad \text{for all } t \in \{0, 1, \ldots, N-1\}.$$

Note that for $\|h\| = 1$

$$\left|\frac{\partial f}{\partial x}\big(t,\phi(t)\big)\big(h(t)\big) - \frac{\partial f}{\partial x}\big(t,\psi(t)\big)\big(h(t)\big)\right| \leqslant \left|\frac{\partial f}{\partial x}\big(t,\phi(t)\big) - \frac{\partial f}{\partial x}\big(t,\psi(t)\big)\right|$$

for all $t \in \{0, 1, \ldots, N-1\}$.

Therefore

$$\|\phi - \psi\| < \delta \quad \Rightarrow \quad \big\|\mathrm{D}F(\phi) - \mathrm{D}F(\psi)\big\| = \sup_{\|h\|=1} \big\|\mathrm{D}F(\phi)h - \mathrm{D}F(\psi)h\big\|$$
$$= \sup_{\|h\|=1}\left(\sup_{t\in\{0,1,\ldots,N-1\}}\left|\frac{\partial f}{\partial x}\big(t,\phi(t)\big)\big(h(t)\big) - \frac{\partial f}{\partial x}\big(t,\psi(t)\big)\big(h(t)\big)\right|\right) < \epsilon.$$

Therefore F is continuously differentiable. □

Theorem 3.1. *Suppose $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible. Then for each ϵ small enough, there is a solution to the boundary value problem* (1)–(2).

Proof. Since $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$ is invertible, then L^{-1} exists and x is a solution of (1)–(2) if and only if $x = \epsilon L^{-1}Fx$. If we define $T : \mathbb{R} \times X \to X$ by $T(\epsilon, x) = x - \epsilon L^{-1}Fx$, then x is a solution of (1)–(2) if and only if $T(\epsilon, x) = 0$. Clearly T is continuously Fréchet differentiable, $T(0,0) = 0$ and $\frac{\partial T}{\partial x}(0,0)$ is the identity map which of course is a bijection. Therefore, by the Implicit Function Theorem, for each ϵ small enough, there exists x_ϵ such that $T(\epsilon, x_\epsilon) = 0$ and thus x_ϵ is a solution of (1)–(2). □

4. The case of singular L

In order to analyze (1)–(2) using the Lyapunov–Schmidt procedure, we now construct projections onto the kernel and image of L.

Definition 4.1. Define $P : X \to X$ by

$$(Px)(t) = S(t)\big(S(0)^T S(0)\big)^{-1} S(0)^T x(0).$$

Proposition 4.1. *P is a projection onto* $\ker(L)$.

Proof. First we must show that $S(0)^T S(0)$ is invertible.

Claim 1. $S(0)^T S(0)$ is invertible.

Proof of Claim: Let $c \in \mathbb{R}^j$ and assume $S(0)^T S(0)c = 0$. Then

$$\begin{aligned}
&c^T S(0)^T S(0)c = 0\\
&\quad\Rightarrow\quad \big(S(0)c\big)^T\big(S(0)c\big) = 0 \quad\Rightarrow\quad \big|S(0)c\big| = 0\\
&\quad\Rightarrow\quad S(0)c = 0 \quad\Rightarrow\quad \Phi(0)b_1c_1 + \cdots + \Phi(0)b_jc_j = 0 \quad \big(c = (c_1, \ldots, c_j)\big)^T\\
&\quad\Rightarrow\quad c_1b_1 + \cdots + c_jb_j = 0 \quad\Rightarrow\quad c_i = 0 \quad \text{for all } i = 1, 2, \ldots, j.
\end{aligned}$$

Therefore $S(0)^T S(0)$ is invertible.

Claim 2. $P^2 = P$.

Proof of Claim: Let $x \in X$.

$$\begin{aligned}
\big(P(Px)\big)(t) &= P\big(S(\cdot)\big(S(0)^T S(0)\big)^{-1} S(0)^T x(0)\big) \\
&= S(t)\big(S(0)^T S(0)\big)^{-1} S(0)^T \big[S(0)\big(S(0)^T S(0)\big)^{-1} S(0)^T x(0)\big] \\
&= S(t)\big(S(0)^T S(0)\big)^{-1} S(0)^T x(0) \\
&= (Px)(t).
\end{aligned}$$

Therefore $P^2 = P$.

Claim 3. $\mathrm{Im}(P) = \mathrm{Ker}(L)$.

Proof of Claim: Let $x \in X$, then

$$(Px)(t) = S(t)\big(S(0)^T S(0)\big)^{-1} S(0)^T x(0) = S(t)\alpha$$

where $\alpha =^{-1} S(0)^T x(0) \in \mathbb{R}^j$. Therefore $\mathrm{Im}(P) \subset \ker(L)$.

Now let $x \in \ker(L)$. Then $x(t) = S(t)\beta$ for some $\beta \in \mathbb{R}^j$ and

$$(Px)(t) = S(t)\big(S(0)^T S(0)\big)^{-1} S(0)^T S(0)\beta = S(t)\beta = x(t).$$

Therefore $x \in \mathrm{Im}(P)$ and thus $\ker(L) \subset \mathrm{Im}(P)$, which implies $\mathrm{Im}(P) = \ker(L)$.

Since P is clearly bounded and linear, then Claims 1–3 verify that P is a projection onto $\ker(L)$. □

4.1. *Projection onto* $\mathrm{Im}(L)$

The following definition and lemmas are vital in the construction of a projection onto the image of L.

Definition 4.2. Assume $\ker((B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))^T) = \mathrm{span}\{c_1, c_2, \ldots, c_j\}$ for some vectors $c_1, c_2, \ldots, c_j \in \mathbb{R}^n$. We define $v_k : \{0, 1, \ldots, N-1\} \to \mathbb{R}^n$ by

$$v_k(t) = \sum_{i=t+1}^{N} \big[B_i \Phi(i)\Phi^{-1}(t+1)\big]^T c_k.$$

Lemma 4.1. *v_k is the zero map if and only if $c_k \in \bigcap_{i=0}^N \ker(B_i^T)$.*

Proof. If v_k is the zero map, then $v_k(t) = 0$ for all $t = 0, 1, \ldots, N-1$. Observe the following telescoping effect:

$$\begin{aligned}
v_k(N-1) = 0 \quad &\Rightarrow \quad B_N^T c_k = 0 \\
v_k(N-2) = 0 \quad &\Rightarrow \quad \big[B_{N-1}\Phi(N-1)\Phi(N-1)^{-1}\big]^T c_k \\
&\qquad + \big[B_N\Phi(N)\Phi(N-1)^{-1}\big]^T c_k = 0 \quad \Rightarrow \quad B_{N-1}^T c_k = 0, \\
&\ \vdots \\
v_k(0) = 0 \quad &\Rightarrow \quad \big[B_1\Phi(1)\Phi(1)^{-1}\big]^T c_k + \cdots + \big[B_N\Phi(N)\Phi(1)^{-1}\big]^T c_k = 0, \\
&\Rightarrow \quad B_1^T c_k = 0.
\end{aligned}$$

Therefore $c_k \in \bigcap_{i=1}^N \ker(B_i^T)$.

However, since $c_k \in \ker((B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))^T)$, then we must also have $c_k \in \bigcap_{i=0}^N \ker(B_i^T)$. The proof in the other direction is trivial. □

Lemma 4.2. *If $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$ for each $k = 1, 2, \ldots, j$, then $\{v_1, v_2, \ldots, v_j\}$ is a linearly independent set.*

Proof. Assume $\sum_{k=1}^j \alpha_k v_k = 0$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_j$. Then we must have $\sum_{k=1}^j \alpha_k v_k(t) = 0$ for all $t = 0, 1, \ldots, N-1$.

Consider

$$\sum_{k=1}^{j} \alpha_k v_k(N-1) = \sum_{k=1}^{j} \alpha_k B_N^T c_k = 0.$$

Now, for all i such that $c_i \notin \ker(B_N^T)$, we must have $\alpha_i = 0$, since whenever $c_i \notin \ker(B_N^T)$, $B_N^T c_i$ is a basis element for the image of the map that sends each $x \in \ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)^T)$ into $B_N^T x$.

If there exists an integer k such that $c_k \in \ker(B_N^T)$, then we continue this process, letting $t = N-2$.

Observe

$$\sum_{k=1}^{j} \alpha_k v_k(N-2) = \sum_{k \text{ such that } c_k \in \ker(B_N^T)} \alpha_k (B_{N-1})^T c_k = 0.$$

For all k such that $c_k \notin \ker(B_{N-1}^T)$, we must have $\alpha_k = 0$ since whenever $c_i \notin \ker(B_{N-1}^T)$, $B_{N-1}^T c_i$ is a basis element for the image of the map that sends each $x \in \ker(B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)^T)$ into $B_{N-1}^T x$.

Now if there are any coefficients, α_k, which we have not yet shown to be 0, then $c_k \in \bigcap_{i=N-1}^N \ker(B_i^T)$ and we continue letting $t = N-3$ and repeat this process if necessary until $t = 0$. If we indeed need to continue until $t = 0$, then there exists an integer k such that $c_k \in \bigcap_{i=2}^N \ker(B_i^T)$. However, for each of these c_k, $c_k \notin \ker(B_1^T)$, for if it were, since $c_k \in \ker((B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))^T)$, we would also have to have $c_k \in \ker(B_0^T)$, which would contradict our hypothesis that $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$. Finally, if we consider

$$\sum_{k=1}^{j} \alpha_k v_k(0) = \sum_{k \text{ such that } c_k \in \bigcap_{i=2}^N \ker(B_i^T)} \alpha_k (B_1)^T c_k = 0,$$

then all the remaining α_k must be 0, by the same argument presented for $t = N-1$ and $t = N-2$.

Since $\alpha_i = 0$ for all $i = 1, \ldots, j$, then $\{v_1, v_2, \ldots, v_j\}$ is a linearly independent set. □

Remark 4.1. For the rest of the paper we will assume that for each k in $\{1, 2, \ldots, j\}$, $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$.

Definition 4.3. $\Psi(t)$ is the $n \times j$ matrix whose ith column is $v_i(t)$.

Proposition 4.2. *If h is in Y, then h is in the image of L if and only if $\sum_{i=0}^{N-1} h^T(i)\Psi(i) = 0$.*

Proof. Let $h \in Y$. From Proposition (3.1) we know h is in the image of L if and only if there exists an element x in X such that

$$B_0 x(0) + B_1\big(\Phi(1)x(0) + \Phi(1)\Phi^{-1}(1)h(0)\big) + \cdots + B_N\left(\Phi(N)x(0) + \Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right) = 0.$$

This holds if and only if

$$B_1\Phi(1)\Phi^{-1}(1)h(0) + B_2\sum_{i=0}^{1}\Phi(2)\Phi^{-1}(i+1)h(i) + \cdots + B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)$$

is in the image of $B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N)$.

Therefore we must have

$$\left(B_1\Phi(1)\Phi^{-1}(1)h(0) + B_2\sum_{i=0}^{1}\Phi(2)\Phi^{-1}(i+1)h(i) + \cdots + B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)\right)^T\beta = 0 \tag{10}$$

for all $\beta \in \ker((B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))^T)$.

Since $\{c_1, c_2, \ldots, c_j\}$ spans $\ker((B_0 + B_1\Phi(1) + \cdots + B_N\Phi(N))^T)$ we see that the above holds if and only if

$$\sum_{i=0}^{N-1} h^T(i)\left(\sum_{m=i+1}^{N}\big[B_m\Phi(m)\Phi^{-1}(i+1)\big]^T\right)c_k = 0, \quad \text{for each } k = 1, 2, \ldots, j, \tag{11}$$

which is equivalent to

$$\sum_{i=0}^{N-1} h^T(i)v_k(i) = 0, \quad \text{for each } k = 1, 2, \ldots, j. \tag{12}$$

Therefore, $h \in \operatorname{Im}(L)$ if and only if $\sum_{i=0}^{N-1} h^T(i)\Psi(i) = 0$, or equivalently $\sum_{i=0}^{N-1}\Psi^T(i)h(i) = 0$. □

We have seen that if $c_k \notin \bigcap_{i=0}^{N}\ker(B_i^T)$ for each $k = 1, 2, \ldots, j$, then $\{v_1, v_2, \ldots, v_j\}$ is a linearly independent set. This allows us to make the following definition.

Definition 4.4. The map $W : Y \to Y$ is defined by

$$(Wh)(t) = \Psi(t)\left(\sum_{i=0}^{N-1}\Psi(i)^T\Psi(i)\right)^{-1}\sum_{i=0}^{N-1}\Psi^T(i)h(i).$$

Proposition 4.3. *The map E given by $E = I - W$ is a projection onto the image of L.*

Proof. We must first show that W is well defined.

Claim 1. $\sum_{i=0}^{N-1}\Psi(i)^T\Psi(i)$ is invertible.

Proof of Claim: Let $\alpha \in \mathbb{R}^j$ and write $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_j)^T$.

Assume $(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i))\alpha = 0$, then

$$\alpha^T \left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right) \alpha = 0 \quad \Rightarrow \quad \sum_{i=0}^{N-1} \big(\Psi(i)\alpha\big)^T \Psi(i)\alpha = 0 \quad \Rightarrow \quad \sum_{i=0}^{N-1} \big|\Psi(i)\alpha\big| = 0.$$

Therefore $\Psi(i)\alpha = 0$ for all $i = 0, 1, \dots, N-1$, and thus

$$v_1(i)\alpha_1 + v_2(i)\alpha_2 + \dots + v_j(i)\alpha_j = 0 \quad \text{for each } i = 0, 1, \dots, N-1.$$

Since $\{v_1, v_2, \dots, v_j\}$ is a linearly independent set, then we must have $\alpha = 0$ and thus $\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)$ is invertible.

Since W is clearly bounded and linear, all we need to show is that W is a projection and $\operatorname{Im}(E) = \operatorname{Im}(L)$.

Claim 2. $W^2 = W$.

Proof of Claim: Let $h \in \mathcal{Y}$,

$$\begin{aligned}
\big(W(Wh)\big)(t) &= W\left(\Psi(\cdot) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \right) \\
&= \Psi(t) \left[\sum_{\alpha=0}^{N-1} \Psi(\alpha)^T \Psi(\alpha) \right]^{-1} \sum_{\beta=0}^{N-1} \Psi(\beta)^T \Psi(\beta) \\
&\quad \times \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^\top h(k) \\
&= \Psi(t) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^\top h(k) = (Wh)(t) \\
\Rightarrow \quad W^2 = W.
\end{aligned}$$

Since W is a projection, then E must also be a projection.

Claim 3. $\operatorname{Im}(E) = \operatorname{Im}(L)$.

Proof of Claim: Let $h \in \mathcal{Y}$,

$$\begin{aligned}
\sum_{i=0}^{N-1} \Psi(i)^T (Eh)(i) &= \sum_{i=0}^{N-1} \Psi(i)^T \big(h(i) - (Wh)(i)\big) \\
&= \sum_{i=0}^{N-1} \Psi(i)^T h(i) \\
&\quad - \sum_{i=0}^{N-1} \Psi(i)^T \left(\Psi(i) \left[\sum_{k=0}^{N-1} \Psi(k)^T \Psi(k) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \right)
\end{aligned}$$

$$= \sum_{i=0}^{N-1} \Psi(i)^T h(i) - \sum_{k=0}^{N-1} \Psi(k)^T h(k) = 0.$$

Therefore $Eh \in \operatorname{Im}(L)$ and thus $\operatorname{Im}(E) \subset \operatorname{Im}(L)$.

Now let $h \in \operatorname{Im}(L)$. Then

$$(Eh)(t) = h(t) - \Psi(t)\left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)\right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) = h(t).$$

Therefore $Eh = h$, and thus $h \in \operatorname{Im}(E)$ which implies $\operatorname{Im}(E) = \operatorname{Im}(L)$. □

A very special case of the projection E appears in [16], where only two-point boundary value problems are considered.

Since P and E are continuous projections, we may now write $X = X_P \oplus X_{I-P}$ and $Y = Y_{I-E} \oplus Y_E$ where $X_P = \operatorname{Im}(P)$, $X_{I-P} = \operatorname{Im}(I-P)$, $Y_E = \operatorname{Im}(E)$, and $Y_{I-E} = \operatorname{Im}(I-E)$.

Lemma 4.3. *The dimension of X_P is the same as the dimension of Y_{I-E}.*

Proof. Since $c_k \notin \bigcap_{i=0}^{N} \ker(B_i^T)$ for each $k = 1, 2, \ldots, j$, then by Lemma 4.2, the span of $\{v_1, v_2, \ldots, v_j\}$ is a j-dimensional space. Let $h \in Y$. Then

$$(Wh)(t) = \Psi(t)(a_1, a_2, \ldots, a_j)^T \quad \text{for some real numbers } a_1, a_2, \ldots, a_j$$
$$\Rightarrow \quad (Wh)(t) = a_1 v_1(t) + a_2 v_2(t) + \cdots + a_j v_j(t) \quad \text{for each } t = 0, 1, \ldots, N-1.$$

Therefore $\operatorname{Im}(W) \subset \operatorname{span}\{v_1, v_2, \ldots, v_j\}$.

Now let $h \in \operatorname{span}\{v_1, v_2, \ldots, v_j\}$. Then $h(t) = \Psi(t)(b_1, b_2, \ldots, b_j)^T$ for some real numbers $b_1, b_2, \ldots, b_j$. Therefore

$$\begin{aligned}(Wh)(t) &= \Psi(t)\left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)\right]^{-1} \sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)(b_1, b_2, \ldots, b_j)^T \\ &= \Psi(t)(b_1, b_2, \ldots, b_j)^T = h(t).\end{aligned}$$

Therefore $Wh = h$ and thus $h \in \operatorname{Im}(W)$ which implies $\operatorname{span}\{v_1, v_2, \ldots, v_j\} = \operatorname{Im}(W)$. Consequently both X_P and Y_{I-E} are j-dimensional spaces. □

Note that $L : X_{I-P} \to \operatorname{Im}(L)$ is a bijection and thus there exists a bounded linear map $M : \operatorname{Im}(L) \to X_{I-P}$ such that:

(i) $LMh = h$ for all $h \in \operatorname{Im}(L)$;
(ii) $MLx = x_{I-P}$ for all $x \in X$.

Definition 4.5. Define $H : \mathbb{R} \times X_P \times X_{I-P} \to Y_{I-E} \times X_{I-P}$ by

$$H(\epsilon, x_P, x_{I-P}) = \begin{pmatrix} WF(x_p + x_{I_P}) \\ x_{I-P} - \epsilon MEF(x_p + x_{I-P}) \end{pmatrix}.$$

Proposition 4.4. *For $\epsilon \neq 0$, $Lx = \epsilon Fx$ if and only if $H(\epsilon, x_P, x_{I-P}) = 0$.*

Proof.

$$\begin{aligned} Lx = \epsilon Fx \quad &\Longleftrightarrow \quad E(Lx - \epsilon Fx) = 0 \quad \text{and} \quad (I-E)(Lx - \epsilon Fx) = 0 \\ &\Longleftrightarrow \quad Lx = \epsilon EFx \text{ and } \epsilon WFx = 0 \\ &\Longleftrightarrow \quad x_{I-P} - \epsilon MEF(x_P + x_{I-P}) = 0 \quad \text{and} \quad WF(x_P + x_{I-P}) = 0 \\ &\Longleftrightarrow \quad H(\epsilon, x_P, x_{I-P}) = 0. \qquad \square \end{aligned}$$

The following result is an immediate consequence of the basic principles of Differential Calculus in Banach Spaces [12].

Proposition 4.5. *H is a continuously Fréchet differentiable map from $\mathbb{R} \times X_P \times X_{I-P}$ into $Y_{I-E} \times X_{I-P}$. If $(\epsilon, x_P, x_{I-P}) \in \mathbb{R} \times X_P \times X_{I-P}$ then for each $(\alpha, p, q) \in \mathbb{R} \times X_P \times X_{I-P}$,*

$$\begin{aligned} &DH(\epsilon, x_P, x_{I-P})(\alpha, p, q) \\ &\quad = \begin{pmatrix} WDF(x_P + x_{I-P})(p+q) \\ q - \alpha MEF(x_P + x_{I-P}) - \epsilon MEDF(x_P + x_{I-P})(p+q) \end{pmatrix}. \end{aligned}$$

Theorem 4.1. *Assume f is continuously differentiable and $c_k \notin \bigcap_{i=0}^{N} \ker(B_i^T)$ for each $k = 1, 2, \ldots, j$. If there exists $\hat{\alpha} \in \mathbb{R}^j$ such that*

$$\sum_{i=0}^{N-1} f\big(i, S(i)\hat{\alpha}\big)^T \Psi(i) = 0 \quad \textit{and} \quad \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}\big(i, S(i)\hat{\alpha}\big) S(i) \quad \textit{is invertible},$$

then for each ϵ small enough, there exists a solution, x_ϵ, to the boundary value problem (1)–(2). *Furthermore,* $\lim_{\epsilon \to 0} \|x_\epsilon - S(\cdot)\hat{\alpha}\| = 0$.

Proof. Let $\hat{x}_P \in X_P$ be given by $\hat{x}_P(t) = S(t)\hat{\alpha}$. Since f is continuously differentiable, then F and H are continuously Fréchet differentiable. Also, since $\sum_{i=0}^{N-1} f(i, \hat{x}_P(i))^T \Psi(i) = 0$, then $F(\hat{x}_P)$ is in the image of L, and thus $H(0, \hat{x}_P, 0) = 0$.

We now show that the partial Fréchet derivative of H with respect to (x_P, x_{I-P}) at $(0, \hat{x}_P, 0)$, $\frac{\partial H}{\partial (x_P, x_{I-P})}(0, \hat{x}_P, 0)$, is a bijection from $X_P \times X_{I-P}$ onto $Y_{I-E} \times X_{I-P}$.

Since

$$DH(\epsilon, x_P, x_{I-P}) : \mathbb{R} \times X_P \times X_{I-P} \to Y_{I-E} \times X_{I-P}$$

is given by

$$\begin{aligned} &DH(\epsilon, x_P, x_{I-P})(\alpha, p, q) \\ &\quad = \begin{pmatrix} WDF(x_P + x_{I-P})(p+q) \\ q - \alpha MEF(x_P + x_{I-P}) - \epsilon MEDF(x_P + x_{I-P})(p+q) \end{pmatrix}, \end{aligned}$$

then

$$\frac{\partial H}{\partial (x_P, x_{I-P})}(0, \hat{x}_P, 0) : X_P \times X_{I-P} \to Y_{I-E} \times X_{I-P}$$

is given by

$$\frac{\partial H}{\partial (x_P, x_{I-P})}(0, \hat{x}_P, 0)(h_1, h_2) = \begin{pmatrix} WDF(\hat{x}_P)(h_1 + h_2) \\ h_2 \end{pmatrix}.$$

We now show that since X_P and Y_{I-E} have the same dimension, then $WDF(\hat{x}_P): X_P \to Y_{I-E}$ a bijection if and only if $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i))S(i)$ is invertible.

Assume $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i))S(i)$ is invertible and let $h \in \ker(WDF(\hat{x}_P))$. Since $h \in X_P$, we can write $h(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}^j$. Since $h \in \ker(WDF(\hat{x}_P))$, then

$$\Psi(t)\left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)\right)^{-1} \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}\big(i, \hat{x}_P(i)\big)h(i) = 0 \quad \text{for all } t = 0, 1, \ldots, N-1.$$

Since the columns of Ψ are linearly independent then

$$\left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)\right)^{-1} \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}\big(i, \hat{x}_P(i)\big)S(i)\alpha = 0,$$

and thus

$$\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}\big(i, \hat{x}_P(i)\big)S(i)\alpha = 0.$$

Since $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i))S(i)$ is invertible, then we must have $\alpha = 0$ and thus $h = 0$.

Now assume $WDF(\hat{x}_P)$ is a bijection and assume there exists a nonzero vector $\alpha \in \mathbb{R}^j$ such that

$$\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}\big(i, \hat{x}_P(i)\big)S(i)\alpha = 0.$$

Let $h(i) = S(i)\alpha$. Then $h \in X_P$, $h \neq 0$ and

$$\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}\big(i, \hat{x}_P(i)\big)h(i) = 0,$$

which implies $WDF(\hat{x}_P)h = 0$. This contradicts the invertibility of $WDF(\hat{x}_P)$, and therefore $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i))S(i)$ is invertible.

Since $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i))S(i)$ is invertible, then we now have that $WDF(\hat{x}_P)$ is a bijection.

Let $(h_1, h_2) \in \ker(\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0))$. Since

$$\begin{pmatrix} WDF(\hat{x}_P)(h_1 + h_2) \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then $h_2 = 0$. Also, since $h_1 + h_2 \in \ker(WDF(\hat{x}_P))$, then $h_1 + h_2 = 0$ and thus $h_1 = 0$.

Therefore $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)$ is one-to-one. Since the dimension of $X_P \times X_{I-P}$ is the same as the dimension of $Y_{I-E} \times X_{I-P}$ then $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)$ is a bijection.

Since H is continuously differentiable, $H(0, \hat{x}_P, 0) = 0$, and $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)$ is a bijection, then by the Implicit Function Theorem, for each ϵ small enough, there exists $(x_P, x_{I-P})_\epsilon$ such that $H(\epsilon, (x_P, x_{I-P})_\epsilon) = 0$. If we let $x_\epsilon = (x_P)_\epsilon + (x_{I-P})_\epsilon$, then $Lx_\epsilon = \epsilon F(x_\epsilon)$ and thus x_ϵ is a solution of (1)–(2). Furthermore, x_ϵ is a continuously differentiable function of ϵ and $\lim_{\epsilon \to 0} \|x_\epsilon - \hat{x}_P\| = 0$. $\square$

5. Example

Consider

$$x(t+1)=\begin{pmatrix}1&0\\1&1\end{pmatrix}x(t)+\epsilon f\bigl(t,x(t)\bigr) \tag{13}$$

subject to

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}x(0)+\begin{pmatrix}0&1\\0&0\end{pmatrix}x(1)+\begin{pmatrix}0&0\\1&0\end{pmatrix}x(N)=0. \tag{14}$$

To put this boundary value problem into our framework we let

$$A(t)=\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad \text{for each } t=0,1,2,\dots,$$

and we let $B_0=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $B_1=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $B_N=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$.

Observe that since $A(t)$ is a constant matrix, then $\Phi(t)=A^t$.

$\ker(B_0+B_1A+B_NA^N)=\ker\left(\begin{smallmatrix}2&2\\1&1\end{smallmatrix}\right)=\operatorname{span}\left\{\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\right\}$, and thus

$$S(t)=\Phi(t)\begin{pmatrix}-1\\1\end{pmatrix}=A^t\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1&0\\t&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-1\\1-t\end{pmatrix}.$$

$\ker((B_0+B_1A+B_NA^N)^T)=\ker\left(\begin{smallmatrix}2&1\\2&1\end{smallmatrix}\right)=\operatorname{span}\left\{\left(\begin{smallmatrix}1\\-2\end{smallmatrix}\right)\right\}$. Note that $\left(\begin{smallmatrix}1\\-2\end{smallmatrix}\right)\notin\ker(B_0^T)$, and thus X_P and Y_{I-E} are both one-dimensional spaces with $\Psi(t)$ given by

$$\Psi(t)=\begin{cases}\left(\begin{smallmatrix}-2\\1\end{smallmatrix}\right) & \text{for } t=0,\\ \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) & \text{for } t=1,2,\dots,N-1.\end{cases}$$

Now assume

$$f(t,x_1,x_2)=\begin{pmatrix}f_1(x_1)\\f_2(t,x_1,x_2)\end{pmatrix}$$

where $f_1:\mathbb{R}\to\mathbb{R}$ and $f_2:\mathbb{R}^3\to\mathbb{R}$ are both continuously differentiable, $f_1(-\alpha)=0$ for some real number α, $f_2(0,x_1,x_2)=0$ and

$$2Nf_1'(-\alpha)+\frac{\partial f_2}{\partial x_1}(0,-\alpha,\alpha)+\frac{\partial f_2}{\partial x_2}(0,-\alpha,\alpha)\neq 0.$$

If we let $\hat{x}_P=S(t)\alpha$, then

$$\begin{aligned}\sum_{k=0}^{N-1} f\bigl(k,\hat{x}_P(k)\bigr)^T\Psi(k)&=\sum_{k=0}^{N-1}\begin{pmatrix}f_1(-\alpha)\\f_2(k,-\alpha,\alpha(1-k))\end{pmatrix}^T\Psi(k)\\&=f_2(0,-\alpha,\alpha)-2Nf_1(-\alpha)=0.\end{aligned}$$

Furthermore, the real number

$$\begin{aligned}&\sum_{i=0}^{N-1}\Psi(i)^T\frac{\partial f}{\partial x}\bigl(i,\hat{x}_P(i)\bigr)S(i)\\&\quad=\sum_{i=0}^{N-1}\Psi(i)^T\frac{\partial f}{\partial x}\bigl(i,-\alpha,\alpha(1-i)\bigr)\begin{pmatrix}-1\\1-i\end{pmatrix}\\&\quad=2Nf_1'(-\alpha)+\frac{\partial f_2}{\partial x_1}(0,-\alpha,\alpha)+\frac{\partial f_2}{\partial x_2}(0,-\alpha,\alpha)\neq 0.\end{aligned}$$

Therefore by Theorem 4.1, for each ϵ small enough there is a unique solution to the boundary value problem (13)–(14). A simple example of a function f satisfying our criteria is given by

$$f(t,x_1,x_2)=\begin{pmatrix} x_1^2+x_1-6 \\ t(x_1^2+x_2^2) \end{pmatrix}.$$

Observe that $f_1(-3)=0$, $f_2(0,x_1,x_2)=0$, and

$$2Nf_1'(-3)+\frac{\partial f_2}{\partial x_1}(0,-3,3)+\frac{\partial f_2}{\partial x_2}(0,-3,3)=-10N\neq 0.$$

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